# **III. Parabolic equations**

We will solve the heat equation.

## **8. Heat transfer with knows temperature at the ends**

We consider the heat transfer phenomenon that is described by the heat equation. The temperature at the ends of the body and at the initial time are given. Using the method of separation of variables, we transform the partial differential equation to two ordinary differential equations, which are connected by a common constant. The spatial equation with boundary conditions is known Sturm–Liouville problem that an infinite set of solutions. Using Fourier series properties and the given initial condition, determine the solution of the initial problem as a sinus Fourier series. The partial case is considered as an example.

### **8.1. Problem statement**

Consider the heat transfer phenomenon for the long thin body. We have the ***heat equation***

 *ut = a2 uxx*, 0 < *x* < *L*, *t* > 0, (8.1)

where *u* is the temperature, *a* is the thermal diffusivity. Suppose the temperature at the ends of the body is zero. Then we have the first order boundary conditions

 *u*(0,*t*) = 0, *u*(*L*,*t*) = 0, *t* > 0. (8.2)

The initial temperature *ϕ* =*ϕ*(*x*) of the body is given. Then we have the initial conditions

 *u*(*x*,0) = *ϕ*(*x*), 0 < *x* < *L*. (8.3)

The system (8.1) – (8.3) is called the ***first boundary problem*** for the heat equation.

### **8.2. Method of separation of variables**

Try to find the solution of the equation (8.1) as a product of the functions of one variables

 *u*(*x*,*t*) = *X*(*x*) *T*(*t*), (8.4)

where the functions *X* and *T* are unknown. We would like to choose it such that the formula (8.4) gives the solution of the considered problem.

Put the function *u* from the equality (8.4) to the heat equation (8.1). We get

*X*(*x*) *T'*(*t*) = *a*2 *X''*(*x*) *T*(*t*),

where *T'* and *X''* are derivatives of the functions *X* and *T* of one variable. Divide this equality by *a*2*XT.* We obtain



We have the equality of the functions of different variables. It can be true only the values at the right hand-side and the left hand-side are constant. Denote this constant by *λ*. We obtain two ordinary differential equations

 *T'*(*t*) = *a*2*λ T*(*t*), *t* > 0, (8.5)

 *X''*(*x*) = *λ X*(*x*), 0 < *x* < *L.*  (8.6)

Thus, our partial differential equation was be transformed to two ordinary differential equations with different independent variable.

Now we put the function *u* from the equality (8.4) to the boundary conditions (8.2). We have

*X*(0) *T*(*t*) = 0, *X*(*L*) *T*(*t*) = 0, *t* > 0.

If *T*(*t*) = 0 is zero for all time, then the function *u* is zero everywhere because of the formula (8.4) that contradicts the initial conditions (8.3). Therefore, we obtain the equalities

 *X*(0) = 0, *X*(*L*) = 0. (8.7)

We have the second order differential equation (8.6) with boundary conditions (8.7).

Of course, the problem (8.6), (8.7) has the trivial zero solution. However, if *X*(*x*) is zero for all *x*, then the function *u* will be zero that contradicts again the initial conditions (5.3). We have the known ***Sturm–Liouville problem***.

### **8.3. Sturm–Liouville problem**

Find the solution of the problem (8.6), (8.7). We have the linear homogeneous second order differential equation

*X''*(*x*) – *λ X*(*x*) = 0.

Using the standard result of the ordinary differential equation theory, we consider the characteristic equation

*z*2 – *λ* = 0.

Then we determine its solution



The result depends from the sign of the constant *λ*. Thus, it is necessary to consider three different cases.

Suppose the constant *λ* is positive. Then the general solution of the equation (8.6) has the exponent form

  (8.8)

where *c*1 and *c*2 are arbitrary. Using the boundary conditions (5.7), we get





We have the system of two linear algebraic equations with respect to the constant *c*1 and *c*2. Determine *c*1 = 0, *c*2 = 0. Therefore, the value *X*(*x*) is zero because of the equality (8.8). However, we would like to determine a non-zero solution of the problem. Hence, this case is not applicable.

Now suppose the constant *λ* is zero. Then the general solution of the equation (8.6) has the linear form

  (8.9)

Using the boundary conditions (8.7), we obtain





This system has zero solution, and the function *X* is zero too. Therefore, the second is not applicable too.

Finally, we suppose the constant *λ* is negative. Then the general solution of the equation (8.6) has the trigonometric form

  (8.10)

Using the boundary conditions (8.7), we get





By first of these equalities, we have



We conclude that one of the multipliers of the value at the left hand-side is zero. If *c*1 = 0, then *X* is zero function because of the equality (8.10). Thus, we determine



This equality can be true, if



Now we determine the infinite family of parameters

  (8.11)

Thus, there exists the infinite set of non-zero solutions of boundary problem (8.6), (8.7). There are the functions

  (8.12)

We use the constant *ck* here, because for *k* we can have the different constant. Any function *Xk*with arbitrary constant *ck* is the solution of the Sturm–Liouville problem.

Now we return to the heat equation.

### **8.4. Solution of the problem (8.1) – (8.3)**

Consider the differential equation (8.5) with parameter *λ* is equal to *λk*. The characteristic equation has the form

*z*– *a*2*λk* = 0.

Then the general solution of the equation (8.5) for the arbitrary *k* is

  (8.13)

where the constants *ak* are arbitrary. Put the values of the functions *Xk* and *Tk* from the equalities (8.12) and (8.13) to the formula (.4). We find the functions

  (8.14)

where the constants  are arbitrary.

The functions *uk* satisfy the vibrating string equation (8.1) and the boundary conditions (8.2) for all values *k* and the constants  The sum of all these solutions satisfies the heat equation, because of its linearity. Then we find

  (8.15)

This function the equation (8.1) and the boundary conditions (8.2). Now it necessary to choose the coefficients  such that the function *u* from the equality (8.15) satisfies the initial conditions (8.2) too.

Put the function *u* from the equality (8.15) to the initial condition (8.3). We get



Thus, we obtain the equalities



This formula give the representation of the function *ϕ* as ***Fourier series***. Using Fourier series theory, find the ***Fourier coefficients***

  (8.16)

Thus, the solution of the first boundary problem for the heat equation is determined by the formula (8.15) with Fourier coefficients (8.16).

Transform this result. Put the coefficients  from the equality (8.16) to the formula (8.15). We get



Determine the function

  (8.17)

that is called the ***Green function***. Then the solution of the problem (8.1) – (8.3) is determined by the formula

  (8.18)

We obtain the direct dependence of the solution of our boundary problem from its initial state.

### **8.5. Analysis of the heat transfer phenomenon**

Consider the partial case of the problem (8.1) – (8.3). Let us analyze the body of the length *L=π* with coefficient *a =* 1. Then we have the heat equation

 *ut = uxx*, 0 < *x* < *π*, *t* > 0. (8.19)

Let the temperature at ends of the body is equal to zero. Then we have the boundary conditions

 *u*(0,*t*) = 0, *u*(*π*,*t*) = 0, *t* > 0. (8.20)

Suppose the initial temperature distribution is determined by the formula *ϕ*(*x*) =sin *x*. Then we have the initial condition

 *u*(*x*,0) = sin *x*, 0 < *x* < *L*. (8.21)

Using the formula (8.15), determine the solution of the problem (8.19) – (8.21) by the formula

  (8.22)

Find the coefficients  by the formulas (8.16). We get



We calculated this integral before. Then we determine the values



Put this result to the formula (8.21). Thus, the solution of the problem (8.19) – (5.21) is

  (8.23)

Check it. Determine











Give the physical interpretation of this result. We have the heat transfer phenomenon. The temperature of the body is zero at the boundary. The initial temperature distribution has the form of sin *x.* By the formula (8.23), the temperature of the body at the arbitrary has the form of sinus too, i.e. its maximum is realized at the middle of the body, and this value decreases if we approach to the boundary. The law of decreasing of the temperature at the middle of the body is determined in the following figure.



Figure 8.1. Decreasing of the temperature at the middle of the body.

Thus, the temperature at the arbitrary point of the body decrease. The temperature distribution at the different time is given in Figure 8.2. Obviously, the temperature decreases by time. Indeed, the middle of the body is hot enough, but we have zero temperature at the ends. Therefore, we observe the heat flux from the middle to the ends. Then the temperature at the middle decrease, because the temperature at the ends is always zero. The heat go out of the body.



Figure 8.2. Temperature distribution.

Suppose now the parameter *a*, i.e. thermal diffusivityis arbitrary. Then the solution of the considered problem will be



This has analogical sense as the formula (8.23). However, the velocity of temperature decreasing depends from *a.* If *a* > 1, then the body conducts heat enough well, and the velocity of temperature decreasing will be greater. If *a* < 1, then the body conducts heat enough bad, and the velocity of temperature decreasing will be smaller.

### **Conclusions**

* The heat transfer phenomenon with given initial and boundary states is described by the first boundary problem for the heat equation.
* The heat equation can be transformed to two ordinary differential equations by the method of separation of variables.
* The spatial ordinary differential equation with boundary conditions, i.e. Sturm–Liouville problem has the infinite set of solutions.
* The heat equation with homogeneous boundary conditions has the infinite set of solutions.
* The solution of the first boundary problem for the heat equation has the representation as a Fourier series.
* The Fourier coefficients of this representation are determined by the initial condition of the considered problem.
* The solution can be determined also by the Green function.
* The temperature of the body decreases over time, and the solution of the problem tends to the equilibrium position.
* The cooling of the body is the faster, the higher the thermal diffusivity.

### Task. **Heat transfer with knows temperature at the ends with fixed ends**

Consider first order boundary problem for the heat equation:

*ut = a2 uxx*, 0 < *x* < *L*, *t* > 0,

*u*(0,*t*) = 0, *u*(*L*,*t*) = 0, *t* > 0,

*u*(*x*,0) = *ϕ*(*x*), 0 < *x* < *L*.

Table of parameters

|  |  |  |  |
| --- | --- | --- | --- |
| Variant | *L* | *a* | *ϕ*(*x*) |
| 1 | π | 1 | 0 |
| 2 | 1 | 2 | - sin π*x* |
| 3 | 2π | 2 | sin (*x/*2) |
| 4 | 2 | ½ | sin 2π*x* |
| 5 | π | ½ | sin *x* |
| 6 | 1 | 1 | 0 |
| 7 | 2π | ½ | 0 |
| 8 | 2 | 2 | 0 |

Task:

1. Find the solution of the problem.
2. Check that this is, in reality the solution.
3. Show the graph (temperature of the body for the different time).
4. Give the physical interpretation of the results.